Tutorial Note VI

1 Exercises about Solving the Heat Equations

Exercise 1.1

Solve the following Cauchy problem:

$$\begin{cases} u_t - u_{xx} = 0; \\ u(0,x) = x^2. \end{cases}$$

Proof. Here we don't intend to use the solution formula to solve the equations. We use a somewhat tricky method to solve it instead. The key observation is that the property that the solution is a polynomial persists. In fact, consider u_{xxx}, then u_{xxx} is a solution to the heat equation. Since $u_{xxx}(0,x) = 0$, $u_{xxx} = 0$. (Here we assume the uniqueness of the Cauchy problem.) Then we have u is a quadratic polynomial at all time. So

$$u(t,x) = A(t)x^2 + B(t)x + C(t).$$

Substituting it into the equations, we have $u(t,x) = x^2 + 2t$. It is easy to check that it is a solution to the Cauchy problem. □

Exercise 1.2

Solve the following initial boundary value problem (IBVP):

$$\begin{cases} u_t - u_{xx} = 0, & (t,x) \in \mathbb{R}^+ \times (0,l); \\ u(t,0) = u(t,l) = 0, & t \in \mathbb{R}^+; \\ u(0,x) = f(x), & x \in [0,l]. \end{cases}$$

Proof. The method is similar to the odd extension. Here we consider odd extensions across two points $(0,0)$ and $(0,l)$. Let

$$\widetilde{f}(x) = \begin{cases} f(x) & x \in [0,l] \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi(x) = \sum_n (\widetilde{f}(2nl + x) - \widetilde{f}(2nl - x)).$$

Note that for each x, there is at most one n such that $f(2nl + x) - f(2nl - x) \neq 0$ (this property is usually called locally finite), so φ is well-defined. It is easy to see that φ is an extension of f and

$$\varphi(x) + \varphi(-x) = 0 \quad \text{and} \quad \varphi(x) + \varphi(2l - x) = 0.$$

If we use φ as an initial data, then the property that the solution is odd symmetric with respect to 0 and l will persist and the boundary condition will be satisfied. Therefore, the solution

$$\begin{aligned} u(t,x) &= \int K(x-y,t) \sum_n (\widetilde{f}(2nl + y) - \widetilde{f}(2nl - y)) \, \mathrm{d}y \\ &= \int_0^l \left[\sum_n (K(x + 2nl - y, t) - K(x - 2nl + y, t)) \right] f(y) \, \mathrm{d}y, \end{aligned}$$

where K is the Gauss kernel. It is easy to check that u is well-defined and is the solution to the IBVP. $\square$

Exercise 1.3

Solve the IBVP:

$$\begin{cases} u_t - u_{xx} = 0 & (t,x) \in \mathbb{R}^+ \times (0, \infty); \\ u(t,0) = h(t) & t \in \mathbb{R}^+; \\ u(0,x) = 0 & x \in (0, \infty). \end{cases}$$

Proof. In Strauss's PDE, it is solved by considering $v = u - h(t)$ and odd extension. Here we use Green's functions for heat equations to solve it. Firstly, we establish the representation formulas for the heat equations. Suppose that u is a solution to the heat equation. In $\overline{\Omega_T}$ where $\Omega_T = \Omega \times (0, T]$, for an arbitrary v, by integration by parts,

$$\int_{\overline{\Omega_T}} v(u_t - \Delta u) = -\int_{\overline{\Omega_T}} u(v_t + \Delta v) + \int_\Omega vu(T) - \int_\Omega vu(0) + \int_{\partial\Omega \times [0,T]} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right).$$

If we make v satisfy

$$\begin{aligned} v_t + \Delta v = \delta \quad & \text{in } \Omega_T; \\ v = 0 \quad & \text{on } \partial\Omega \times (0, T], \end{aligned}$$

then v is called a Green's function for the heat equation. For the region $(0, \infty) \times (0, T]$, similar to the Laplacian equations, the Green function $G(x, T; y, s)$ is

$$K(x, y; T, s) - K(-x, y; T, s).$$

Slightly lifting the time of the Green function, that is, letting

$$v(y,s) = K(x,y;T+\varepsilon,s) - K(-x,y;T+\varepsilon,s),$$

we have

$$\int_{(0,\infty)} vu(T) = \int_{(0,\infty)} vu(0) - \int_{(0,T]} u\frac{\partial v}{\partial n}.$$

Letting $\varepsilon \to 0$, we have

$$u(x,T) = -\frac{1}{\sqrt{4\pi}}\int_0^T \frac{x}{(T-t)^{3/2}} \mathrm{e}^{-\frac{x^2}{4(T-t)}} h(t).$$

And it could be verified that u is a solution to the IBVP. □

Remark 1.1

Exercise 1 is from Strauss's PDE exercise 2.4.9, exercise 2 is from John's PDE 7.1.(c), and exercise 3 is from John's PDE 7.1.(c) problem 2.

2 Uniqueness of the Cauchy Problems

Generally speaking, there is no uniqueness for Cauchy problem (in fact, for IBVPs in unbounded domains). However, if we impose boundedness conditions, uniqueness will follow.

Theorem 2.1 *There is at most one solution to the problem:*

$$\begin{cases} u_t - u_{xx} = f; \\ u(0,x) = g; \\ |u(t,x)| \le A\mathrm{e}^{a|x|^2}. \end{cases}$$

Proof. It suffices to establish a maximum principle in the following form: if u satisfies the heat equation and $u(t,x) \le A\mathrm{e}^{a|x|^2}$, then

$$\sup_{t,x} u(t,x) \le \sup_x u(0,x).$$

Take T such that $8Ta < 1$. It suffices to.prove

$$\sup_{t\in[0,T],x} u(t,x) \le \sup_x u(0,x)$$

since we could repeat it to obtain the complete maximum principle. Fix x_0, and let

$$u_\varepsilon(t,x) = u(t,x) - \varepsilon(2T-t)^{-1/2}\mathrm{e}^{\frac{|x-x_0|^2}{4(2T-t)}}.$$

Consider the domain $[0,T] \times B(x_0,\rho)$ and apply the maximum principle for bounded domains, then we have

$$u_\varepsilon(t, x_0) \le \sup_{\Gamma_T} u_\varepsilon,$$

where $\Gamma_T = \overline{B(x_0,\rho)_T} \setminus B(x_0,\rho)_T$ is the parabolic boundary. On $[0,T] \times \partial B(x_0,\rho)$,

$$u_\varepsilon(t,x) \le A\mathrm{e}^{a|x|^2} - \varepsilon(2T-t)^{-1/2}\mathrm{e}^{\frac{\rho^2}{4(2T-t)}} \le A\mathrm{e}^{a(\rho+|x_0|)^2} - \varepsilon(2T)^{-1/2}\mathrm{e}^{\frac{\rho^2}{8T}}.$$

Since $1/8T > a$, if ρ is large enough, $u_\varepsilon \le \sup_x u(0,x)$ on $[0,T] \times \partial B(x_0,\rho)$. Moreover, it is clear that

$$u_\varepsilon(0,x) \le u(0,x).$$

So

$$u_\varepsilon(t,x_0) \le \sup_x u(0,x).$$

It follows that

$$u(t,x_0) \le \varepsilon(2T-t)^{-1/2} + \sup_x u(0,x).$$

Letting $\varepsilon \to 0$, we have

$$u(t,x_0) \le \sup_x u(0,x).$$

So we complete the proof. $\square$